

# Detecting magnetic CMB polarization on an incomplete sky

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The full sky cosmic microwave background polarization field can be decomposed into ‘electric’ and ‘magnetic’ components. Working in harmonic space we construct magnetic variables that can be measured from observations over only a portion of the sky. Our construction is exact for azimuthally symmetric patches, but should continue to perform well for arbitrary patches. For isotropic, uncorrelated noise the variables have a very simple diagonal noise correlation, and further analysis using them should be no harder than analysing the temperature field. We estimate the tensor mode amplitude that could be detected by the Planck satellite and discuss the sensitivity of future experiments in the presence of a contaminating weak lensing signal.

Observations of fluctuations in the cosmic microwave background (CMB) provide a wealth of information about primordial inhomogeneities in the universe. One of the most interesting questions is whether there was a tensor (gravitational wave) component, as predicted by simple inflationary models. CMB polarization measurements offer a unique probe of this signal [1, 2, 3].

Polarization of the cosmic microwave sky is produced by electron scattering, as photons decouple from the primordial plasma. Gravitational waves produce ‘magnetic’ (i.e. curl) and ‘electric’ (i.e. gradient) polarization components at a comparable level by anisotropic redshifting of the energy of photons. Magnetic polarization is not produced by density perturbations, so detection of a magnetic component would provide strong direct evidence for the presence of a primordial gravitational wave (tensor) component.

The problem is that the primordial polarization is only observable over the region of the sky that is not contaminated by emission from our galaxy and other foreground sources of polarization. But the electric/magnetic decomposition is inherently *non-local*, and *non-unique* in the presence of boundaries. To do an exact, lossless separation on the incomplete sky one would need to know the boundary conditions exactly—both the polarization *and its derivative* around the edge of the observed region. Clearly this is not feasible with noisy observed data.

However the hypothesis of a magnetic signal can still be tested. One possibility which avoids differentiating the observed data is to construct line integrals of the polarization as discussed in Refs. [4, 5]. The problem with these line integrals is that there are an infinite number of them, and they are not statistically independent. One would therefore prefer a set of magnetic variables to which the electric component does not contribute, but which are finite in number and statistically independent (at least for idealized noise properties). In this letter we shall construct such variables explicitly for circular sky patches. A more complete exposition of our method using the spin-weight formalism is given in Ref. [6].

The observable polarization field is described in terms of the two Stokes’ parameters  $Q$  and  $U$  with respect to a particular choice of axes about each direction on the sky. We use spherical polar coordinates, and the Stokes’ parameters define a symmetric and trace-free (STF) rank two linear polarization tensor on the sphere

$$\mathcal{P}^{ab} = \frac{1}{2}[Q(\hat{\theta} \otimes \hat{\theta} - \hat{\phi} \otimes \hat{\phi}) - U(\hat{\theta} \otimes \hat{\phi} + \hat{\phi} \otimes \hat{\theta})]. \quad (1)$$

A two dimensional STF tensor can be written as a sum of ‘gradient’ and ‘curl’ parts

$$\mathcal{P}_{ab} = \nabla_{\langle a} \nabla_{b \rangle} P_E - \epsilon^c_{\langle a} \nabla_{b \rangle} \nabla_c P_B. \quad (2)$$

where  $\nabla$  is the covariant derivative on the sphere, angle brackets denote the STF part on the enclosed indices, and round brackets denote symmetrization. The underlying scalar fields  $P_E$  and  $P_B$  describe electric and magnetic polarization respectively and are clearly non-local functions of the Stokes’ parameters. One can define scalar quantities which are local in the polarization by taking two covariant derivatives to form  $\nabla^a \nabla^b \mathcal{P}_{ab} = (\nabla^2 + 2)\nabla^2 P_E$  and  $\epsilon^b_c \nabla^c \nabla^a \mathcal{P}_{ab} = (\nabla^2 + 2)\nabla^2 P_B$  which depend only on the electric and magnetic polarization respectively. Here we focus on the magnetic component, integrating the latter equation in order to construct an observable that depends only on the magnetic component of  $\mathcal{P}_{ab}$ , but does not involve its derivatives. We start by defining the surface integral

$$B'_W \equiv -2 \int_S dS W^* \epsilon^b_c \nabla^c \nabla^a \mathcal{P}_{ab}, \quad (3)$$

where  $W$  is a window function defined over some patch  $S$  of the observed portion of the sky. The factor of minus two is included to make our definition equivalent to that in Ref. [6]. Integrating by parts we have

$$B'_W = \sqrt{2} \int_S dS W_B^{ab*} \mathcal{P}_{ab} - 2 \oint_{\partial S} dl^a (\epsilon^b_a W^* \nabla^c \mathcal{P}_{cb} - \epsilon^b_c \nabla^c W^* \mathcal{P}_{ab}), \quad (4)$$

where  $W_{B\,ab} \equiv \sqrt{2}\epsilon^c{}_{(a}\nabla_{b)}\nabla_c W$  is an STF tensor window function.

We choose window functions so that the line integrals that appear in the construction of  $B'_W$  contain no contribution from the electric polarization. In general this requires that  $W$  and  $\nabla_a W$  vanish on the boundary  $\partial S$ . The surface integral

$$B_W \equiv \sqrt{2} \int_S dS W_B^{ab*} \mathcal{P}_{ab} \quad (5)$$

then depends only on the magnetic polarization.

We expand the window functions in spherical harmonics as

$$W = \sum_{l \geq 2} \sum_{|m| \leq l} \sqrt{\frac{(l-2)!}{(l+2)!}} W_{lm} Y_{lm}. \quad (6)$$

(The square root factor is included for later convenience.) We need not include  $l = 0$  and  $1$  spherical harmonics since they do not contribute to the tensor window functions. In practice, we are only interested in probing scales to some particular  $l_{\max}$  (the magnetic signal from tensor modes has maximal power for  $l \approx 100$  and decreases rapidly with  $l$ ), so the sum in Eq. (6) can be truncated at some finite  $l_{\max}$ .

The polarization tensor  $\mathcal{P}_{ab}$  can be expanded over the whole sky in terms of STF tensor harmonics

$$\mathcal{P}_{ab} = \frac{1}{\sqrt{2}} \sum_{lm} \left( E_{lm} Y_{(lm)ab}^G + B_{lm} Y_{(lm)ab}^C \right), \quad (7)$$

where  $Y_{(lm)ab}^G$  and  $Y_{(lm)ab}^C$  are the gradient and curl tensor harmonics of opposite parities defined in Ref. [1]. From the orthogonality of the spherical harmonics over the full sphere it follows that

$$B_{lm} = \sqrt{2} \int_{4\pi} dS Y_{(lm)}^{C\,ab*} \mathcal{P}_{ab}. \quad (8)$$

In a rotationally-invariant ensemble, the expectation values of the harmonic coefficients define the power spectrum:

$$\langle B_{l'm'}^* B_{lm} \rangle = \delta_{l'l} \delta_{m'm} C_l^{BB}. \quad (9)$$

The form of the harmonic expansion (6) of the window function ensures that the tensor window is

$$W_{B\,ab} = \sum_{lm} W_{lm} Y_{(lm)ab}^C, \quad (10)$$

where the sum is over  $l \geq 2$  and  $|m| \leq l$ . Evaluating the surface integrals in Eq. (5) we find

$$B_W = \sum_{lm} W_{lm}^* \tilde{B}_{lm}, \quad (11)$$

where the pseudo-multipoles  $\tilde{B}_{lm}$  are obtained by restricting the integral in Eq. (8) to the region  $S$ :

$$\tilde{B}_{lm} = \sum_{l'm'} (W_{+(lm)(lm)'} B_{l'm'} - i W_{-(lm)(lm)'} E_{l'm'}) \quad (12)$$

and the coupling matrices are given by

$$W_{+(lm)(lm)'} \equiv \int_S dS Y_{(lm)ab}^{C*} Y_{(lm)'}^{C\,ab} \quad (13)$$

$$W_{-(lm)(lm)'} \equiv i \int_S dS Y_{(lm)ab}^{C*} Y_{(lm)'}^{G\,ab}. \quad (14)$$

The matrix  $W_{-(lm)(lm)'}$  controls the contamination with electric polarization and can always be written as a line integral around the boundary of  $S$ . Our aim is to construct window functions  $W_{lm}$  that remove this contamination. For azimuthally-symmetric patches the coupling matrices are block diagonal ( $W_{\pm(lm)(lm)'} \propto \delta_{mm'}$ ), and  $W_{-(lm)(lm)'}$  has only two non-zero eigenvalues for  $|m| \geq 2$ . For  $m = 0$  and  $|m| = 1$  there are zero and one non-zero eigenvalues respectively.

We now give a practical method for constructing a non-redundant set of window functions  $\{W_I\}$ , where  $I$  labels the particular window, that achieve exact separation for azimuthal patches. The corresponding (cleanly separated) magnetic observables will be denoted  $B_{W_I}$ . Vectors are now denoted by bold Roman font, e.g.  $\mathbf{B}_W$  has components  $B_{W_I}$ , and  $\mathbf{B}$  has components  $B_{lm}$ , and matrices are denoted by bold italic font, e.g.  $\mathbf{W}_{\pm}$  have components  $W_{\pm(lm)(lm)'}$ . We present the method in a form that is applicable (though no longer exact) to arbitrary shaped regions  $S$ .

In matrix form we start with the observed data vector

$$\tilde{\mathbf{B}} = \mathbf{W}_+ \mathbf{B} - i \mathbf{W}_- \mathbf{E}. \quad (15)$$

The essence of our method is to choose a window function  $\mathbf{W}$  so as to project out the range of  $\mathbf{W}_-$  corresponding to its non-zero eigenvalues.

We first diagonalize  $\mathbf{W}_+$  so it can be written  $\mathbf{W}_+ = \mathbf{U}_+ \mathbf{D}_+ \mathbf{U}_+^\dagger$ . This allows us to identify the linear combinations  $\mathbf{U}_+^\dagger \mathbf{B}$  that are poorly determined by  $\tilde{\mathbf{B}}$ —those corresponding to the small diagonal elements of  $\mathbf{D}_+$ . These correspond to polarization patterns that are badly supported on the observed patch of the sky and need to be removed from the analysis to construct a non-redundant set of window functions. The distribution of eigenvalues of  $\mathbf{W}_+$  is bimodal and the exact definition of ‘small’ is not critical [6]. We define an operator  $\tilde{\mathbf{U}}_+$  which projects onto the eigenvectors of  $\mathbf{W}_+$  whose eigenvalues are close to one. This amounts to removing the appropriate columns of  $\mathbf{U}_+$  to give the column orthogonal matrix  $\tilde{\mathbf{U}}_+$ . The corresponding smaller square diagonal matrix is  $\tilde{\mathbf{D}}_+$ , and we have

$$\tilde{\mathbf{U}}_+^\dagger \tilde{\mathbf{B}} \approx \tilde{\mathbf{D}}_+ \tilde{\mathbf{U}}_+^\dagger \mathbf{B} - i \tilde{\mathbf{U}}_+^\dagger \mathbf{W}_- \mathbf{E}. \quad (16)$$

We now multiply by  $\tilde{\mathbf{D}}_+^{-1/2}$ , defined by  $[\tilde{\mathbf{D}}_+^{-1/2}]_{ij} \equiv \delta_{ij}[\tilde{\mathbf{D}}_+]_{ii}^{-1/2}$ , to give

$$\tilde{\mathbf{D}}_+^{-1/2}\tilde{\mathbf{U}}_+^\dagger\tilde{\mathbf{B}} \approx \tilde{\mathbf{D}}_+^{1/2}\tilde{\mathbf{U}}_+^\dagger\mathbf{B} - i\tilde{\mathbf{D}}_+^{-1/2}\tilde{\mathbf{U}}_+^\dagger\mathbf{W}_-\mathbf{E}. \quad (17)$$

This step ensures that isotropic noise in the map gives isotropic noise on the variables  $\tilde{\mathbf{D}}_+^{-1/2}\tilde{\mathbf{U}}_+^\dagger\tilde{\mathbf{B}}$ , as shown in Ref. [6].

We now project out the unwanted  $\mathbf{E}$  contamination by projecting out of the range of  $\tilde{\mathbf{D}}_+^{-1/2}\tilde{\mathbf{U}}_+^\dagger\mathbf{W}_-$ . The singular-value decomposition for a non-square matrix takes the form  $\tilde{\mathbf{D}}_+^{-1/2}\tilde{\mathbf{U}}_+^\dagger\mathbf{W}_- = \mathbf{U}\mathbf{D}\mathbf{V}^\dagger$  where  $\mathbf{D}$  is diagonal matrix,  $\mathbf{U}$  is unitary and  $\mathbf{V}$  is a column orthogonal rectangular matrix. There are at most two non-zero singular values (diagonal elements of  $\mathbf{D}$ ) per  $m$ , and the corresponding left singular vectors (columns of  $\mathbf{U}$ ) form an orthonormal basis for the range of  $\tilde{\mathbf{D}}_+^{-1/2}\tilde{\mathbf{U}}_+^\dagger\mathbf{W}_-$ . We can project out of this range by defining  $\tilde{\mathbf{U}}$  as the matrix obtained by removing the columns of  $\mathbf{U}$  where the corresponding singular value is non-zero. Thus, choosing the window functions

$$\mathbf{W}^* = \tilde{\mathbf{U}}^\dagger\tilde{\mathbf{D}}_+^{-1/2}\tilde{\mathbf{U}}_+^\dagger, \quad (18)$$

we obtain the vector of separated magnetic observables

$$\mathbf{B}_W = \mathbf{W}^*\tilde{\mathbf{B}} \equiv \tilde{\mathbf{U}}^\dagger\tilde{\mathbf{D}}_+^{-1/2}\tilde{\mathbf{U}}_+^\dagger\tilde{\mathbf{B}} \approx \tilde{\mathbf{U}}^\dagger\tilde{\mathbf{D}}_+^{1/2}\tilde{\mathbf{U}}_+^\dagger\mathbf{B}. \quad (19)$$

For azimuthal patches the separation is exact; the approximation sign arises only from our use of  $\mathbf{W}_+ \approx \tilde{\mathbf{U}}_+\tilde{\mathbf{D}}_+\tilde{\mathbf{U}}_+^\dagger$  in simplifying the matrix that premultiplies  $\mathbf{B}$ .

In the case of idealized, un-correlated isotropic noise on the Stokes' parameters the covariance of the noise  $\Delta\mathbf{B}_W$  on  $\mathbf{B}_W$  is simply [6]

$$\mathbf{N} \equiv \langle \Delta\mathbf{B}_W \Delta\mathbf{B}_W^\dagger \rangle = \sigma_N^2 \mathbf{I}, \quad (20)$$

where  $\sigma_N^2$  is the reciprocal weight per solid angle on the Stokes' parameters.

We now consider how to test the null hypothesis that the magnetic signal is due entirely to noise. If the noise is Gaussian the  $\mathbf{B}_W$  will be Gaussian and the simplest thing to do is a  $\chi^2$  test by computing  $\chi^2 = \mathbf{B}_W^\dagger \mathbf{N}^{-1} \mathbf{B}_W$  (for isotropic noise this is just  $\chi^2 = \mathbf{B}_W^\dagger \mathbf{B}_W / \sigma_N^2$ ). Whilst the CMB magnetic polarization signal is expected to be Gaussian, any spurious or unexpected signal may not be. One may therefore also wish to do a more sophisticated set of statistical tests at this point.

If the signal is Gaussian and we know the expected shape of the power spectrum we can use the statistic [6]

$$\nu' \equiv \frac{\mathbf{B}_W^\dagger \mathbf{N}^{-1} \mathbf{S} \mathbf{N}^{-1} \mathbf{B}_W - \text{tr}(\mathbf{N}^{-1} \mathbf{S})}{\sqrt{4\mathbf{B}_W^\dagger \mathbf{N}^{-1} \mathbf{S} \mathbf{N}^{-1} \mathbf{S} \mathbf{N}^{-1} \mathbf{B}_W - 2\text{tr}(\mathbf{N}^{-1} \mathbf{S} \mathbf{N}^{-1} \mathbf{S})}}. \quad (21)$$

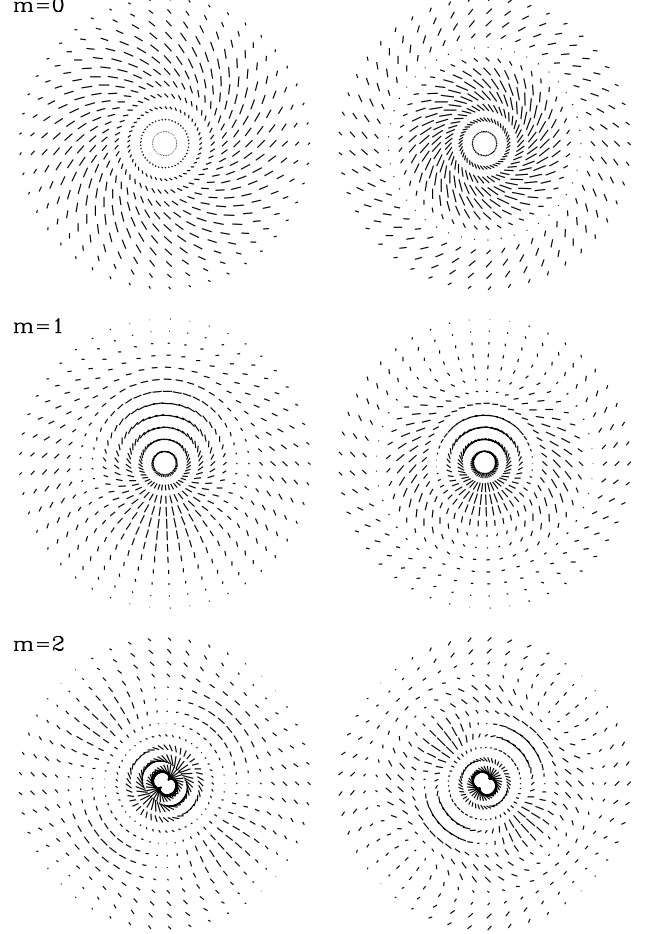


FIG. 1: The real space window functions for an azimuthally symmetric sky patch with  $\theta < 10^\circ$ , evaluated in the frame where the signal is diagonal so the leftmost window produces the largest signal for that  $m$ .

giving the number of standard deviations of the maximum likelihood from zero for small signal to noise. Here the theory covariance is

$$\mathbf{S} \equiv \langle \mathbf{B}_W \mathbf{B}_W^\dagger \rangle = \tilde{\mathbf{U}}^\dagger \tilde{\mathbf{D}}_+^{1/2} \tilde{\mathbf{U}}_+^\dagger \mathbf{C}^{BB} \tilde{\mathbf{U}}_+ \tilde{\mathbf{D}}_+^{1/2} \tilde{\mathbf{U}}, \quad (22)$$

where the diagonal magnetic power spectrum matrix is given by  $[\mathbf{C}^{BB}]_{(lm)(l'm')} = \delta_{mm'} \delta_{ll'} C_l^{BB}$ . The  $\nu'$  distribution is computed by Monte-Carlo simulation in order to assign robust probabilities [6].

For a particular theoretical model we can rotate to the frame where the signal matrix is diagonal, giving rotated  $\mathbf{B}_W^{(R)}$  that are fully statistically independent. In Fig. 1 we plot the window functions for the  $\mathbf{B}_W^{(R)}$  which give the largest contributions to the signal for a typical flat scale-invariant  $\Lambda$ CDM model. The window functions are plotted as line segments of length  $\sqrt{Q_W^2 + U_W^2}$  at angle  $\tan^{-1}(U_W/Q_W)/2$  to the  $\theta$  direction where the real

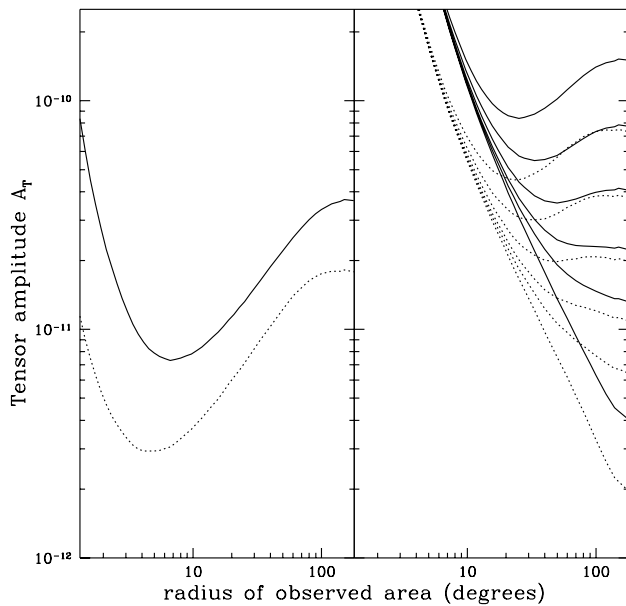


FIG. 2: The smallest gravitational wave amplitude  $A_T$  (defined as in Ref. [7]) that could be detected at 99 per cent confidence with probability 0.95 (solid lines) and 0.5 (dotted lines) by a one year survey that maps a circular patch of sky of a given radius assuming uniform noise. The left panel is for detector sensitivity  $s = 10\mu\text{K}\sqrt{\text{sec}}$  (about three times better than Planck) and no lensing signal, the right panel is for sensitivities (bottom to top)  $s^2 = \{0, 25, 50, 100, 200, 400\} \mu\text{K}^2\text{sec}$  and a magnetic lensing signal with white spectrum  $C_{\text{lens}}^{BB} = 4.4 \times 10^{-6} \mu\text{K}^2$ .

quantities  $Q_W$  and  $U_W$  are defined so that

$$\Re B_W^{(R)} = \int_S dS (Q_W Q + U_W U). \quad (23)$$

Of the current funded experiments, only Planck is likely to detect magnetic polarization if the levels are as predicted by standard cosmological models. As a toy model we consider the 143 and 217 GHz polarized channels of the Planck High Frequency Instrument. We approximate the noise as isotropic and ignore the finite beam width. Combining maps from these two channels with inverse variance weighting, we find  $\sigma_N \approx 6 \times 10^{-3} \mu\text{K}/\text{K}$ , where  $Q$  and  $U$  are expressed as dimensionless thermodynamic equivalent temperatures in units of the CMB temperature. We apply an azimuthally-symmetric galactic cut of 20 degrees either side of the equator and use  $l_{\text{max}} = 250$ . We find that the null-buster would give a detection at 99 per cent significance for a tensor initial power spectrum amplitude (defined as in Ref. [7])  $A_T \gtrsim 4 \times 10^{-10}$  with probability greater than 90 per cent. This corresponds to about 1/10 of the large scale  $C_l$  detected by COBE, and would be generated by inflationary models with slow-roll parameter  $\epsilon \gtrsim 0.01$  and

energy scale  $V^{1/4} \gtrsim 2 \times 10^{16} \text{ GeV}$  at horizon crossing (for example the simplest  $\phi^2$  inflation model).

Considering experiments more sensitive than Planck, one must also take into account the magnetic signal produced by weak lensing of the much larger electric polarization [8, 9]. We model this lensing signal as an additional constant Gaussian white noise. In Fig. 2 we show the minimum value of  $A_T$  that could be detected at 99 per cent significance as a function of the radius of the observed patch, assuming a flat  $\Lambda\text{CDM}$  model which reionizes at  $z = 6.5$  (as evidenced by recent Keck observations). More sensitive surveys require larger patches in order to distinguish the tensor signal from the lensing signal, and there is an optimal survey size when the instrument noise is approximately equal to the lensing noise. Even with no instrument noise the lensing signal places a lower limit on what can be detected, and in this limit one should survey as much of the sky as possible. Since we have assumed that source subtraction and component separation can be done exactly our results are probably optimistic.

In summary, we have shown how to construct a set of statistically independent magnetic polarization observables from observations covering only a portion of the sky. For large patches these contain most of the available information (see Ref. [6] for a quantitative discussion). Some information loss is inevitable as one cannot do lossless clean separation without taking derivatives of the polarization on the boundary. The variables have simple noise properties and can be constructed easily using our harmonic-based approach that automatically removes redundancy due to the finite sky coverage.

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